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Publication date:
1999

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Abstract

A model of an oligopolistic market with a homogeneous product is examined. Each subject of the model uses a conjecture about the market response to variations of its production volume. The conjecture value depends upon both the current total volume of production at the market and the subject's contribution into it. Under general enough assumptions, the equilibrium existence and uniqueness theorems are proven. Furthermore, a particular assumption – namely, constant elasticity, – is considered, and the generalized Stackelberg model comprising several leaders is investigated.

JEL-code: D2.

Keywords: oligopolistic market, conjectural variations, equilibrium, leaders and followers.

[*]This research was supported by the Russian Foundation of Basic Research (RFBR), Grant No. 96-15-98624

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1 Introduction

In this paper, some well-known models of the oligopolistic market with a homogeneous product have been extended to the case when the producers, instead of the standard Cournot model assumptions, use the more general ones:

$$G_i(\eta) = G + (\eta - q_i)w_i(G; q_i): \tag{1}$$

Here G is the current total volume of the product sold at the market, q_i and η are the current and expected supplies of the i–th producer, resp.,, whereas $G_i(\eta)$ is the total market volume conjectured by the i-th agent as a response to changing his (or her) own supply from q_i to η. The conjecture function w_i will be referred to as the i-th agent's influence quotient. The standard Cournot model assumes $w_i \equiv 1$ for all i.

There exists a plethora of works on imperfect competition (cf. [1] – [9]) in which the questions of the rationality of the assumptions are discussed. But the purpose of our paper is somewhat different. We aim at finding some very general forms of possible conjectures that allow an equilibrium to exist. The main difference of our framework from those of the above cited papers consists in that we permit the conjectures being functions of two variables (unlike in [3], [5] where they are functions of one variable). Moreover, we do not demand that the influence quotients be differentiable or even continuous; the agents need not to be identical (unlike [13]).

Another essential motivation for developing these results was a desire to formally embed the Stackelberg model into the generalized Cournot framework, at least in terms of the first order necessary conditions. This embedding is realized through the explicit construction of the (one or multiple) leaders' influence quotients. Since a leading agent of the Stackelberg model takes into account the other agents' optimal responses, the leader's conjecture inevitably loses its smoothness if one of the followers stops producing. That leads not only to rejecting the previously assumed continuity of factors w_i in (1) , but also to the splitting of conjecture (1) into two parts. More precisely, we allow the factors w_i to take different values for $\eta > q_i$ and $\eta < q_i$. These extentions lead to modifying some other assumptions, too.

The paper is organized as follows. Section 2 presents the extended problem specification, describes and partially discusses assumptions concerning functions relevant in the model, and justifies the equilibrium notion. Section 3 is dedicated to the equilibrium existence theorem, wereas Section 4 deals with theorems stating the uniqueness of the equilibrium total volume. In Section 5, a particular case is examined, which comprises a version of the model in which each agent supposes the elasticity of the total market volume, with respect to his output, to remain constant. Section 6 deals with the particular case of influence quotients being functions of the total bargain volume only. In Section 7, the Stackelberg model (more precisely, the one extended here) is embedded formally into the generalized Cournot model. The embedding is realized in terms of the first-order optimality conditions. Section 8 provides some comparative statics analysis for the generalized Cournot and Stackelberg models. Section 9 provides some examples of all the model types mentioned above, and compares them to each other. Section 10 contains conclusions.

2 Problem Specification

Consider n firms producing a homogeneous product and denote by q_i the i-th firm current output and by $f_i(q_i)$ its cost function, $i = 1, \ldots, n$. Let G be the total market output volume, and $p(G)$ be the inverse demand function value, i.e. the price of the product unit established at the market with the total bargain volume G. We also allow a constant volume Q of the extraneous supply of the product. Therefore, the balanced market implies the equality $G = Q + \sum q_i$.

As mentioned above, instead of (1), we use the conjectures

$$G_i(\eta) = \begin{cases} G + (\eta - q_i)w_i^+(G, q_i), & \eta > q_i, \\ G + (\eta - q_i)w_i^-(G, q_i), & \eta < q_i. \end{cases} \tag{2}$$

From this onward, for brevity purposes, we omit the explicit indication of the fact that $G_i(\eta)$ is a function of G and q_i, too. Behaviour of each subject is determined by its current state (G, q_i) and conjecture (2). Namely, if the i-th agent produces η, then its expected profit is equal to

$$\pi_i(\eta) = \eta p(G_i(\eta)) - f_i(\eta), \tag{3}$$

and he chooses his optimal output volume maximizing function (3).

Since this paper deals only with static problems, we are interested in the situations where each of the expected profit functions (3) attains its (local) maximum at $\eta = q_i$. If the market is balanced, such a situation is naturally treated as an equilibrium. In order not to avoid considering the interesting particular case of Section 5, we assume the existence of conjectures (2) only for situations with $G > 0$, $q_i > 0$. The expected profit function (3) is also defined in this area. Nevertheless, under certain conditions, some agents of the market can have zero outputs. Because of that, the equilibrium will be defined in terms of the first order necessary conditions for the expected profit function (3) to attain its maximum. The latter conditions are then extended to the agents with $q_i = 0$ in a natural manner.

Before the formal problem specification, we state the basic assumptions concerning the cost functions f_i, the conjecture derivatives $w_i^{\pm}$ and the inverse demand function p.

A1. Each of the functions $f_i(q_i)$ defined over $q_i \geq 0$, are continuously differentiable, non-decreasing and convex.

A2. The inverse demand function $p(G)$ defined for positive G is continuously differentiable, $p(G) > 0$ and $p'(G) < 0$ for all G.

A3. For each i there exists $H_i > 0$ such that $f_i'(H_i) = p(H_i)$.

A4. For each i, the functions $w_i^{\pm}(G; q_i)$ are defined for $0 < q_i \le G$, and the products $q_i w_i^{\pm}$ do not decrease over $q_i > 0$. Furthermore, at the domain $(G; q_i) > 0$, the function w_i^+ is upper semicontinuous (u.s.c) with respect to G and continuous from the right with respect to q_i; w_i^- is lower semicontinuous (l.s.c.) by G and continuous from the left by q_i, and the following relationships take place:

$$0 \le w_i^-(G; q_i) \le w_i^+(G; q_i); \tag{4}$$

and

$$w_i^+(G; G) = 1. \tag{5}$$

A5. Principle of Potential Participation. For some k there exist $G_0 > 0$ and $q_{0k} > 0$ such that the relationship $G < G_0$ implies the inequality

$$p(G) + p'(G)G - f_k'(q_{0k}) > 0. \tag{6}$$

A6. The function $p(G)G$ is concave with respect to G.

Assumption A4 needs some explanations. Equality $G = q_i$ means that the i-th agent produces the whole output in a monopolistic manner, and condition (5) reflects that. In the classical Cournot model, $w_i^-(G; q_i) = w_i^+(G; q_i) = 1$ for all $(G; q_i)$, which indicates the absence of the other agents' reaction to the i-th subject's output variations. If $w_i^{\pm}(G; q_i) > 1$, then the i-th agent is expecting the others' total output to increase or decrease simultaneously with its own product volume. Moreover, the rate of the expected reaction grows together with the value $w_i^{\pm}$. Therefore, in this case, the right inequality in (4) signifies that the rate of the reaction expected by the i-th agent to an increase in its output is not lower than that to a decrease in it. In case of $w_i^{\pm}(G; q_i) < 1$, the situation is reverse. The others' common output variation is opposite to that of i-th subject, and the rate of the former is growing when the function value $w_i^{\pm}$ diminishes. However, non-negativity of the latter values implies the shift of the total production volume to the same direction as that of the i-th agent. At last, the right inequality in (4) indicates that the market abandoned by the i-th subject is conquered with the rate equal or greater than that of giving it up when seized by the same agent.

Monotonicity of the products $q_i w_i^{\pm}(G; q_i)$ by q_i can also be interpreted in economic terms. Consider the following identities

$$\varepsilon(G; q_i; \eta) = \frac{\partial \ln G_i(\eta)}{\partial \ln \eta} = \frac{\eta}{G_i(\eta)} \cdot \frac{\partial G_i(\eta)}{\partial \eta}.$$

The value $\varepsilon(G; q_i; \eta)$ indicates expected elasticity of the total product volume with respect to the i-th subject's output. For $\eta = q_i \pm 0$ one obtains $\varepsilon(G; q_i; q_i \pm 0) = q_i w_i^{\pm}(G; q_i)/G$. Therefore, the assumed monotonicity of $q_i w_i^{\pm}$ by q_i for any fixed value G means that the expected (by the i-th subject) elasticity of the total output does not decrease along with the subject's contribution increasing. In addition, this monotonicity together with condition (5) imply the inequality $w_i^+(G; q_i) \le G/q_i$ for $0 < q_i < G$.

Now consider the functions $\sigma_i^+(G;q_i) = q_i w_i^+(G;q_i)/G$, $\sigma_i^-(G;q_i) = q_i w_i^-(G;q_i)/G$. Since they are monotone by q_i and non-negative, we can define their limit values as follows:

$$\sigma_i^+(G;0) = \lim_{q_i \to 0} \sigma_i^+(G;q_i); \quad \sigma_i^-(G;0) = \lim_{q_i \to 0} \sigma_i^-(G;q_i).$$

The functions $\sigma_i^{\pm}(G;q_i)$ have a clear economic meaning. Namely, under conjecture (2), $\sigma_i^+(G;q_i)$ presents the expected (by the i–th agent) elasticity of the total market output with respect to its own output q_i when the latter increases. The function $\sigma_i^-(G;q_i)$ can be treated in a similar way when q_i decreases. In the elasticity terms, assumption A4 allows the following re-formulation.

A4$'$. For each i, functions $\sigma_i^{\pm}(G;q_i)$ are defined for $G > 0$, $0 \le q_i \le G$, and do not decrease with respect to q_i. Furthermore, for $q_i > 0$, function $\sigma_i^+(G;q_i)$ is u.s.c. by G and continuous from the right with respect to q_i, function $\sigma_i^-(G;q_i)$ is l.s.c. by G and continuous from the left with respect to q_i, and the following relationships are valid

$$0 \le \sigma_i^-(G;q_i) \le \sigma_i^+(G;q_i) \le \sigma_i^+(G;G) = 1. \tag{7}$$

The assumed here monotonicity by q_i signifies that at a fixed total market volume, each subject expects his influence to grow (more precisely, not to decrease) along with his contribution increase. The right equality in (7), similarly to (5), indicates that the subject does not expect the other suppliers to arise when he's increasing his output after having seized the market completely.

Principle A5 means that for G sufficiently small (and hence, p sufficiently high) there exists at least one agent striving to increase his output volume over q_{0k}.

Differentiating the expected profit function (3) by η, one obtains the relationships

$$\pi_i'(\eta) = \begin{cases} p(G_i(\eta)) + \eta w_i^+(G;q_i)p'(G_i(\eta)) - f_i'(\eta); & \eta > q_i, \\ p(G_i(\eta)) + \eta w_i^-(G;q_i)p'(G_i(\eta)) - f_i'(\eta): & \eta < q_i. \end{cases} \tag{8}$$

At the point $\eta = q_i$, the function $\pi_i(\eta)$ has the leftside derivative $\pi_i^-(G;q_i)$ and the rightside derivative $\pi_i^+(G;q_i)$ calculated by the formulae

$$\pi_i^+(G;q_i) = p(G) + \sigma_i^+(G;q_i)p'(G)G - f_i'(q_i); \tag{9}$$

$$\pi_i^-(G;q_i) = p(G) + \sigma_i^-(G;q_i)p'(G)G - f_i'(q_i). \tag{10}$$

Using the first order necessary local maximum conditions for each subject's expected profit function, we specify our problem as follows. Given $Q \ge 0$, find a vector $Z = (G; q_1; \ldots; q_n) \in R_+^{n+1}$ such that the balance equality holds

$$\sum q_i + Q = G; \tag{11}$$

and for each i, the following inequalities are valid:

$$\pi_i^+(G, q_i) = p(G) + \sigma_i^+(G, q_i)p'(G)G - f_i'(q_i) \leq 0; \qquad (12)$$

and if $q_i > 0$, then

$$\pi_i^-(G, q_i) = p(G) + \sigma_i^-(G, q_i)p'(G)G - f_i'(q_i) \geq 0; \qquad (13)$$

If $\sigma_i^+(G, q_i) = \sigma_i^-(G, q_i)$ and $q_i > 0$, inequalities (12) and (13) are tantamount to a single equality.

In order to treat a solution to problem (12)–(13) as an equilibrium for the agents with $q_i > 0$, we need showing concavity of each subject's expected profit function.

Theorem 1. Under assumptions A1 – A6, for each agent with $q_i > 0$, his expected profit function $\pi_i(\eta)$ is concave over $\eta \geq 0$.

Proof. In order to prove the theorem, it suffices to verify that both righthand sides in (8) do not increase by η and that $\pi_i^+(G, q_i) \leq \pi_i^-(G, q_i)$. The latter inequality follows from (7) and A2 immediately. As for the righthand sides in (8), we show the upper one not to increase. The same property of the lower one can be demonstrated in a similar manner.

Consider $q_i \leq \eta_1 < \eta_2$. Since the value $w_i^+(G, q_i)$ is non-negative, it follows from (2) that $G \leq G_i(\eta_1) \leq G_i(\eta_2)$. Hence, making use of A2, we get $p(G_i(\eta_1)) \geq p(G_i(\eta_2))$. Moreover, A1 implies $f_i'(\eta_1) \leq f_i'(\eta_2)$. In addition, if $p'(G_i(\eta_1)) \geq p'(G_i(\eta_2))$, then $\eta_1 p'(G_i(\eta_1)) \geq \eta_2 p'(G_i(\eta_2))$ due to the negativity of p'. All the above inequalities taken together imply $\pi_i'(\eta_2) \leq \pi_i'(\eta_1)$.

Otherwise, if $p'(G_i(\eta_1)) < p'(G_i(\eta_2))$, we use (2) and re-arrange the difference $\pi_i'(\eta_1) - \pi_i'(\eta_2)$ in the form

$$\begin{aligned}\pi_i'(\eta_1) - \pi_i'(\eta_2) &= [p(G_i(\eta_1)) - p(G_i(\eta_2))] + [G_i(\eta_1)p'(G_i(\eta_1)) - \\ &- G_i(\eta_2)p'(G_i(\eta_2))] + \left\{\left(1 - \sigma_i^+(G, q_i)\right) G\,[p'(G_i(\eta_2)) - p'(G_i(\eta_1))]\right\} + \\ &+ [f_i'(\eta_2) - f_i'(\eta_1)].\end{aligned}$$

It is obvious from A6 that the sum of expressions in two former square brackets takes non-negative values, and the last term is non-negative due to the convexity of f_i. The sign of the term in the curly brackets depends on that of the factor $\left(1 - \sigma_i^+(G, q_i)\right)$ which is non-negative according to (7). The proof of the theorem is complete. ■

In what follows we justify the definition of equilibrium from the point of view of an agent with $q_i = 0$. The maximum necessary and sufficient conditions (12)–(13) for the expected profit function can be re-written in the form of the pair of inequalities

$$-\sigma_i^-(G, q_i)p'(G)G \leq p(G) - f_i'(q_i) \leq -\sigma_i^+(G, q_i)p'(G)G. \qquad (14)$$

The concluding righthand term in (14) can be treated as a barrier that must be overhauled by the difference between the price and the cost of production of an extra product unit (i.e. the marginal cost of production) for the i-th agent to motivate his output increase. The beginning lefthand term is interpreted in a similar manner.

Since the concluding righthand term in (14) tends to the limit value $-\sigma_i^+(G;0)p'(G)G$ as q_i vanishes, it is natural enough to postulate the following behaviour of the agent with $q_i = 0$. Namely, he starts producing whenever the difference between the price $p(g)$ and $f_i'(0)$ exceeds that limit barrier, i.e. the inequality $p(G) - f_i'(0) > -\sigma_i^+(G;0)p'(G)G$ holds. However, condition (12) means quite the opposite at $q_i = 0$. Therefore, the agent i has no reason to shift from zero when (12) takes place.

Notice that if conjecture (2) is valid for $q_i = 0$ too, i.e. if the value $w_i^+(G;0)$ is well defined, then continuity of w_i^+ from the right implies $\sigma_i^+(G;0) = 0$. Then it follows from (12) that $p(G) \leq f_i'(0)$, and again there is no reason for the i-th subject to start producing.

3 Existence Theorem

Assumptions A4 and A4$'$ postulate only one-sided continuity of functions $w_i^\pm$ and $\sigma_i^\pm$ with respect to q_i for fixed G and their semicontinuity with respect to G for fixed q_i. However, when proving the existence theorem, we need limit transitions with respect to the variable pair $(G;q_i)$, including the case $q_i \to 0$. To avoid the complicated discourses in the main theorem, we obtain the lemmas stated below. First we extend the domain of function $w_i^\pm$ having accepted $w_i^\pm(G;q_i) = w_i^\pm(G;G) = 1$ for $q_i > G > 0$. Then continuity and monotonicity assumptions from A4 are obviously not violated. The domain of functions $\sigma_i^\pm(G;q_i)$ and $\mu_i^\pm(G;q_i)$ is extended respectively without breaking assumption A4$'$.

Lemma 1. Functions $\mu_i^+(G;q_i)$ are continuous from the right with respect to q_i and lower semicontinuous by G for $q_i \geq 0$, as well as functions $\mu_i^-(G;q_i)$ are continuous from the left with respect to q_i and upper semicontinuous by G for $q_i > 0$.

Proof. The respective continuity of functions $\mu_i^\pm$ by q_i and semicontinuity by G at $q_i > 0$ follow immediately from A1, A2 and A4 (A4$'$). As for the lower semicontinuity of $\mu_i^+(G;q_i)$ at $q_i = 0$, it is equivalent to the upper semicontinuity of the function $\sigma_i^+(G;0)$ by G due to the same assumptions. However, the latter feature does take place because of the obvious equality

$$\sigma_i^+(G;0) = \inf_{q_i>0} \sigma_i^+(G;q_i)$$

and the function $\sigma_i^+(G;q_i)$ upper semicontinuity by G for any $q_i > 0$. This completes the proof. ∎

Lemma 2. Under assumptions A4 (A4$'$) functions $\mu_i^+;\ \mu_i^-$, $i = 1;\dots;n$, satisfy the following equalities:

$$\liminf_{(G;q_i)\to(\bar G;\bar q_i)} \mu_i^+(G;q_i) = \mu_i^+(\bar G;\bar q_i); \quad \forall \bar G > 0;\ \bar q_i \geq 0;$$

$$\limsup_{(G;q_i)\to(G;\bar q_i)} \mu_i^-(G;q_i) = \mu_i^-(G;\bar q_i); \quad \forall G > 0;\ \bar q_i > 0:$$

Proof. Let us fix a positive G and a non-negative $\bar q_i$ and consider an arbitrary sequence of points $(G^k;q_i^k) > 0$ convergent to $(G;\bar q_i)$. For an arbitrary $\varepsilon > 0$, there exists a number K such that $q_i^k \le \bar q_i + \varepsilon$ for each $k > K$. The function $\mu_i^+(G;q_i)$ being monotonically nonincreasing by q_i (as it follows from A1 and A4$'$), we obtain $\mu_i^+(G^k;q_i^k) \ge \mu_i^+(G^k;\bar q_i + \varepsilon); \quad \forall k > K$: Since $\mu_i^+(G;q_i)$ is l.s.c. by G for positive q_i, the latter inequality yields

$$\liminf \mu_i^+(G^k;q_i^k) \ge \liminf \mu_i^+(G^k;\bar q_i + \varepsilon) \ge \mu_i^+(G;\bar q_i + \varepsilon): \tag{15}$$

From the right-side continuity of $\mu_i^+(G;q_i)$ by q_i, we have the relationship $\lim_{\varepsilon\to+0} \mu_i^+(G;\bar q_i + \varepsilon) = \mu_i^+(G;\bar q_i)$. Hence, inequalities (15) imply (as $\varepsilon \to +0$)

$$\liminf \mu_i^+(G^k;q_i^k) \ge \mu_i^+(G;\bar q_i): \tag{16}$$

Since the sequence $(G^k;q_i^k)$ was selected arbitrarily, (16) implies

$$\mu_i^+(G;\bar q_i) = \lim_{q_i\to\bar q_i+0} \mu_i^+(G;q_i) \ge \liminf_{(G;q_i)\to(G;\bar q_i)} \mu_i^+(G;q_i) \ge \mu_i^+(G;\bar q_i): \tag{17}$$

The first assertion of the Lemma follows from inequalities (17). The second equality is obtained in a similar way, thus completing the proof. ■

Theorem 2. Let $Q > 0$ and assumptions A1 – A4 hold. Then problem (11)–(13) has a solution. In addition, if assumption A5 is valid, then the problem is solvable in case $Q = 0$, too.

Proof. For each i, we construct a multivalued mapping $G \mapsto [q_i^-;q_i^+]$ that associates to each $G > 0$ a segment, the endpoints of which are defined as follows:

$$q_i^- = \inf\left\{q_i : 0 \le q_i \le H_i : \mu_i^+(G;q_i) \le 0\right\}; \tag{18}$$

$$q_i^+ = \sup\left\{q_i : 0 \le q_i \le H_i : \mu_i^-(G;q_i) \ge 0\right\}; \tag{19}$$

here $H_i > 0$ is the scalar from condition A3. Moreover, we put $q_i^+ = 0$ and/or $q_i^- = H_i$, if one of the sets (18), (19) is empty.

Now we prove that $q_i^- \le q_i^+$. On the contrary, suppose that $q_i^- > q_i^+$. Then $\mu_i^-(G;q_i^-) < 0$ and $q_i^- > 0$. In view of Lemma 1, the functions $\mu_i^-(G;q_i)$ are continuous from the left by q_i, hence $\mu_i^-(G;q_i^- - \delta) < 0$. Since $\mu_i^-(G;q_i) \ge \mu_i^+(G;q_i)$ for each pair $(G;q_i)$, it then follows that $\mu_i^+(G;q_i^- - \delta) < 0$, too. The latter inequality contradicts the definition of q_i^-. Therefore, $q_i^- \le q_i^+$.

If both the sets subject to infimum and supremum in (18) and (19) are non-empty, then Lemma 1 and monotonicity of functions $\mu_i^\pm(G;q_i)$ imply that all the points of the closed segment $[q_i^-;q_i^+]$ satisfy inequalities (12) and (13). If the set in (19) is empty, then $\mu_i^+(G;q_i) \le \mu_i^-(G;q_i) < 0$ for all $0 \le q_i \le H_i$, i.e. $q_i^- = q_i^+ = 0$, and $q_i = 0$ satisfies

(12). On the other hand, if the set in (18) is empty, then $\mu_i^-(G;q_i) \geq \mu_i^+(G;q_i) > 0$ for all $0 \leq q_i \leq H_i$, i.e. $q_i^- = q_i^+ = H_i$. Thus we have shown that in each case except for $q_i^- = q_i^+ = H_i$, all the points of the segment $[q_i^-;q_i^+]$ solve problem (12)–(13).

Suppose that $Q > 0$. Consider the point-to-set mapping $G \mapsto [q_i^-;q_i^+]$ on the half-line $[Q;+\infty)$ and demonstrate it to be closed. Since the mapping's graph is the intersection of the epigraph of the function $q_i^-(G)$ and the undergraph of the function $q_i^+(G)$ (i.e. the set of points $\{(G;\lambda) : G > 0;\ \lambda \leq q_i^+(G)\}$), it suffices to establish the two sets' closedness. In view of their symmetric construction, we do that only for the epigraph of the function $q_i^-(G)$. Let a sequence $\left\{(G^k;q_i^k)\right\}$ with $G^k \geq Q$, $q_i^k \geq q_i^-(G)$ be convergent to a point $(\bar{G};\bar{q}_i)$. Two outcomes are possible. If there exisits an infinite subsequence with $q_i^k = H_i$, then $\bar{q}_i = H_i$ and the point $(\bar{G};\bar{q}_i)$ obviously belongs to the epigraph. Otherwise, if $q_i^k < H_i$ from some k onward, then $q_i^-(G^k) < H_i$ which implies $\mu_i^+(G^k;q_i^-(G^k)) \leq 0$. Since $\mu_i^+(G;q_i)$ is monotone with respect to q_i, then $\mu_i^+(G^k;q_i^k) \leq 0$, too. In view of Lemma 2, we obtain the limit relationship $\mu_i^+(\bar{G};\bar{q}_i) \leq 0$, which means that $(\bar{G};\bar{q}_i)$ belongs to the epigraph of the function $q_i^-(G)$. The closedness of the undergraph of the function $q_i^+(G)$ is established in a similar manner. Thus the closedness of the mapping $G \mapsto [q_i^-;q_i^+]$ is verified.

Now consider the sum of above constructed mappings and the value Q, i.e. the multi-valued mapping that associates to each $G > 0$ the segment $A(G) = [Q + \sum q_i^-; Q + \sum q_i^+]$. We examine this mapping on the segment $[Q; Q + \sum H_i]$. It is easily checked that all the conditions of the Kakutani theorem are satisfied in this case, so there exists a fixed point $G \in A(G)$. In other words, there is a $(n+1)$-tuple $Z = (G;q_1;\dots;q_n)$ satisfying the balance equality (11) and the restrictions $q_i^-(G) \leq q_i \leq q_i^+(G)$, $i = 1;\dots;n$. Now we demonstrate that $q_i^-(G) < H_i$ holds. Indeed, if $q_i^-(G) = H_i$ for some i, then $q_i = H_i$, hence (11) and $Q > 0$ imply $G > H_i$. However, the latter along with A3 yields $p(G) - f_i'(H_i) < 0$ which means that $\mu_i^-(G;H_i) < 0$. Since $\mu_i^-(G;q_i)$ is continuous by q_i from the left, we come to $q_i^+(G) < H_i = q_i^-(G)$ which contradicts the above obtained estimate $q_i^- \leq q_i^+$. Thus, the inequality $q_i^-(G) < H_i$ really holds, whence q_i solves problem (12)–(13) and the whole $(n+1)$–tuple $Z = (G;q_1;\dots;q_n)$ solves problem (11)–(13). This completes the proof in case $Q > 0$.

Now let assumption A5 be valid. For each $Q > 0$, fix up some $(n+1)$–tuple $Z(Q) = [G(Q);q_1(Q);\dots;q_n(Q)]$ solving problem (11)–(13), and direct Q to zero. Since the estimates $Q \leq G(Q) \leq Q + \sum H_i$, $0 \leq q_i < H_i$ hold, there exists a cluster point $\bar{Z} = (\bar{G};\bar{q}_1;\dots;\bar{q}_n)$. Suppose that $\bar{G} = 0$. Then a positive Q exists such that $G(Q) < G^0$ and $q_k(Q) < q_{0k}$ which is prohibited by A5. Hence $\bar{G} > 0$, and we can use Lemma 2 implying that the $(n+1)$–tuple $\bar{Z}$ solves problem (11)–(13) in case $Q = 0$. The theorem is proven completely. ■

Remark 1. In the particular case when $w_i^- = w_i^+ \equiv 1$, assumptions A1 — A3 guarantee the equilibrium existence.

4 Uniqueness of Solution

The question of the equilibrium uniqueness can be divided into two parts. First, from the general modeling point of view, it is interesting to examine uniqueness of the equilibrium total market bargain volume G. This aspect is thouroughly investigated in the section. However in the general case, at the equlilibrium state, that total bargain volume can be distributed among the active producers in various proportions. This question is connected with the uniqueness of solutions to the local problems (12)–(13) and discussed briefly in the end of the section.

To examine conditions ensuring uniqueness of the solution to problem (11)–(13), we keep assumptions A1 – A6 intact and add the following one.

A7. For each i, relationships $0 < G_1 < G_2$ and $0 < \delta < 1$ imply the inequality

$$w_i^+(G_1; \delta G_1) \le w_i^-(G_2; \delta G_2). \tag{20}$$

Moreover,

$$w_i^-(G; q_i) < \frac{G}{q_i}, \quad \text{if} \quad 0 < q_i < G. \tag{21}$$

Notice that condition (20) reflects the co-ordinated non-decreasing of functions w_i^+ and w_i^- along the rays emitting from the origin and contained in the positive orthant. The condition can be re-written also as $\sigma_i^+(G_1; \delta G_1) \le \sigma_i^-(G_2; \delta G_2)$ which implies the limit relationship $\sigma_i^+(G_1; 0) \le \sigma_i^-(G_2; 0)$ when δ vanishes. Thus, under assumption A7, the inequality

$$\sigma_i^+(G_1; \delta G_1) \le \sigma_i^-(G_2; \delta G_2) \tag{22}$$

is valid for $\delta \ge 0$.

Definition 1. Let $Z(Q) = [G(Q); q_1(Q); \ldots; q_n(Q)]$ be an equilibrium. We call it non-monopolistic, if $q_i < G$, $i = 1; \ldots; n$.

Theorem 3. Under assumptions A1 – A7, for each $Q \ge 0$, the non-monopolistic equilibrium total bargain value $G(Q)$ is determined uniquely.

Proof. For a given $Q \ge 0$, on the contrary, suppose that there exist two non-monopolistic equilibria

$$Z_1(Q) = [G_1(Q); q_1^1(Q); \ldots; q_n^1(Q)];$$

$$Z_2(Q) = [G_2(Q); q_1^2(Q); \ldots; q_n^2(Q)]$$

with $G_1 = G_1(Q) < G_2(Q) = G_2$. Define

$$I(G_1) = \{1 \le i \le n : q_i^1(Q) > 0\};$$

$$I(G_2) = \{1 \le i \le n : q_i^2(Q) > 0\}.$$

Observe that $G_2 > Q$ and consequently, $I(G_2) \neq \emptyset$. Now we prove that for each $i \in I(G_2)$ the following inequality holds

$$\frac{q_i^2(Q)}{G_2} < \frac{q_i^1(Q)}{G_1}. \tag{23}$$

Indeed, denote by $q_1 = q_i^1(Q)$ and $q_2 = q_i^2(Q)$ and suppose, on the contrary, that $q_2/G_2 \geq q_1/G_1$. Then obviously $q_2 > q_1$ and there exists $\bar{q}$ such that $q_1 \leq \bar{q} \leq q_2$ and $\bar{q}/G_2 = q_1/G_1 = \delta$. Using (22) and A4$'$, we obtain

$$0 \leq z_1 = \sigma_i^+(G_1, q_1) \leq \sigma_i^-(G_2, \bar{q}) \leq \sigma_i^-(G_2, q_2) = z_2.$$

Since $Z_2(Q)$ is a non-monopolistic equilibrium, then $q_2 < G_2$, and condition (21) implies $z_2 < 1$. Moreover, relationships (12),(13) for this i yield

$$p(G_1) + G_1 p'(G_1) z_1 - f_i'(q_1) \leq p(G_2) + G_2 p'(G_2) z_2 - f_i'(q_2).$$

Convexity of f_i allows one to omit the last terms in both parts without ceasing the inequality to be valid:

$$p(G_1) + G_1 p'(G_1) z_1 \leq p(G_2) + G_2 p'(G_2) z_2. \tag{24}$$

If $z_1 = 0$, then (24) contradicts decreasing of $p(G)$. Let $z_1 > 0$. Subtract from (24) the inequality $(1 - z_2)p(G_1) > (1 - z_2)p(G_2)$, which takes place because of A2 and $z_2 < 1$, and obtain the relationship $z_2 p(G_1) + G_1 p'(G_1) z_1 < z_2 p(G_2) + G_2 p'(G_2) z_2$. Taking into account that $z_2 \geq z_1 > 0$ and $p'(G_1) < 0$, we deduce from the latter the following inequality

$$p(G_1) + G_1 p'(G_1) < p(G_2) + G_2 p'(G_2).$$

Since $G_1 < G_2$, the last inequality contradicts A6. Thus, we have proven that (23) holds for $i \in I(G_2)$.

Inequality (23) shows that $I(G_2) \subseteq I(G_1)$. Consequently, the balance equalities for Z_1 and Z_2 imply the following series of relationships

$$1 = \sum_{i \in I(G_2)} \frac{q_i^2(Q)}{G_2} + \frac{Q}{G_2} < \sum_{i \in I(G_2)} \frac{q_i^1(Q)}{G_1} + \frac{Q}{G_1}$$
$$\leq \sum_{i \in I(G_1)} \frac{q_i^1(Q)}{G_1} + \frac{Q}{G_1} = 1,$$

which leads to an impossible inequality $1 < 1$. It means that the assumption about existence of two non-monopolistic equilibria with different total bargain volumes was wrong, which completes the proof of the Theorem. ■

If $Q > 0$, then each equilibrium is non-monopolistic. In this case, according to Theorem 3, the value of equilibrium total bargain volume $G(Q)$ is unique. However if $Q = 0$, both monopolistic and non-monopolistic equilibria with different total bargain values may occur in the framework of a single problem. For instance, that situation is possible when functions f_i and $\eta(G) = p(G)G$ are piece-wise linear. Nevertheless, the following result obtains:

Corollary 1. In addition to Theorem 3 assumptions, suppose that for each i either f_i is strictly convex, or for all $G > 0$ the relationships $0 \le q_i^1 < q_i^2$ imply the inequality

$$\sigma_i^+(G; q_i^1) < \sigma_i^-(G; q_i^2). \tag{25}$$

Then the non-monopolostic equilibrium vector $Z(Q)$ is unique.

Proof. On the contrary, suppose that for an equilibrium total bargain volume G and for some $i \in I(G)$, there exist certain values $0 < q_i^1 < q_i^2$ satisfying inequalities (12) and (13). Then use either strict convexity of f_i or inequality (25) and write down the series of relationships

$$\begin{aligned} 0 \le \mu_i^-(G; q_i^2) &= p(G) + \sigma_i^-(G; q_i^2)p'(G)G - f_i'(q_i^2) < \\ &< p(G) + \sigma_i^+(G; q_i^1)p'(G)G - f_i'(q_i^1) = \mu_i^+(G; q_i^1) \le 0, \end{aligned}$$

which implies an impossible inequality $0 < 0$. This contradiction completes the proof. ■

Remark 2. Under assumptions of Corollary and with $Q > 0$, the (non-monopolistic) equilibrium exisits uniquely. In order to get a similar result in case of $Q = 0$, we need specifying the form of the influence coefficients $w_i^{\pm}$, as it is done in Sections 5 and 6.

Remark 3. Monopolistic equilibria (when $Q = 0$) can be found out by maximizing functions $\mu_k(G; G) = p(G)G - f_k(G)$ along the half-axis $(0, +\infty)$ under conditions $\mu_i^+(G; 0) \le 0$ for $i \neq k$.

Remark 4. In the particular case when $w_i^+ = w_i^- \equiv 1$, assumptions A1–A3 and functions f_i' being monotone increasing suffice to prove the existence of a unique equilibrium.

5 Case of Constant Elasticity

In this Section, we discuss in more details the particular case when the functions f_i and p are twice continuously differentiable, and functions w_i^+ and w_i^- coincide over all the positive orthant $(G, q_i) > 0$ except for the diagonal ray $q_i = G,\ q_i > 0$, and they assume the following form:

$$w_i^-(G; q_i) = w_i^+(G; q_i) = \alpha_i + \sigma_i \frac{G}{q_i}, \quad \text{if } 0 < q_i < G; \tag{26}$$

$$w_i^-(G; G) = \alpha_i + \sigma_i; \qquad w_i^+(G; G) = 1. \tag{27}$$

Here $0 \le \sigma_i < 1$; $0 \le \alpha_i \le 1 - \sigma_i$. It is easily verified that these w_i^- and w_i^+ satisfy conditions A4 and A7. Therefore, Theorems 1 and 2 are valid for problem (11)–(13) with these functions. As p is twice differentiable, condition A6 reduces to the inequality

$$2p'(G) + p''(G)G \le 0; \qquad \forall G > 0. \tag{28}$$

Moreover, we replace condition (25) from Corollary 1 by the following (a bit stronger) assumption.

A8. Functions f_i and p are twice continuously differentiable, and for each i either $f_i''(q_i) > 0$ for all $q_i > 0$ and $\lim\limits_{q_i \to +\infty} f_i'(q_i) = +\infty$, or $\alpha_i > 0$.

The case $\alpha_i = 0$ corresponds to a constant elasticity of the expected total bargain volume G variation with respect to that of q_i. Lemma 3 proven below is an instrument for investigating the generalized Stackelberg model. Since the case $Q > 0$ is the only relevant at that, the value $w_i^+(G; G) = \infty$ will not be actually used.

Under assumption A8 along with f_i and p being twice differentiable, examine the behaviour of solutions $q_i = q_i(G)$ to problem (12)–(13) which reduces to the following form: find a $q_i \ge 0$ satisfying conditions

$$q_i = 0; \qquad \text{if} \quad f_i'(0) - \sigma_i p'(G)G - p(G) \ge 0; \tag{29}$$

and

$$f_i'(q_i) - \alpha_i q_i p'(G) - \sigma_i p'(G)G - p(G) = 0; \tag{30}$$

if $q_i > 0$. Extend functions f_i smoothly to the negative half-axis $(-\infty; 0)$, having put $f_i(q_i) = f_i(0) + f_i'(0)q_i + f_i''(0)q_i^2/2$ at the points $q_i < 0$. Under assumptions A1–A8, solutions $\tilde{q}_i = \tilde{q}_i(G)$ of equation (30) are determined uniquely and continuously differentiable for all $G > 0$. If $\tilde{q}_i(G) > 0$ at $G > 0$, then the function $q_i(G)$ coincides with $\tilde{q}_i(G)$. Otherwise, if $\tilde{q}_i(\bar{G}_i) = 0$ at some $\bar{G}_i$, i.e. $f_i'(0) = \sigma_i p'(\bar{G}_i)\bar{G}_i + p(\bar{G}_i)$, then due to $0 \le \sigma_i < 1$ and assumptions A2, A6, we have the inequality $f_i'(0) > \sigma_i p'(G)G + p(G)$ hold at each $G > \bar{G}_i$. Therefore, $\tilde{q}_i(G) < 0$ for such G whereas $q_i(G) = 0$. Thus, we have proven that $q_i(G) = \max\{\tilde{q}_i(G); 0\}$, and functions $q_i(G)$ are continuously differentiable everywhere except for the single point $\bar{G}_i$. At the point $\bar{G}_i$, the function $q_i(G)$ has one-sided derivatives $q_i'(\bar{G}_i + 0) = 0$ and $q_i'(\bar{G}_i - 0) \le 0$. Derivatives of the functions $q_i(G)$ at the points $G < \bar{G}_i$ can take either positive or negative values. However, we demonstrate below that at the points G satisfying the inequality

$$Q + \sum q_i(G) \le G; \tag{31}$$

(for instance, at an equilibrium), there is at most one subject with the positive derivative value. Define

$$I(G) = \{i : q_i(G) > 0\}; \qquad I^+(G) = \{i \in I(G) : q_i'(G) > 0\}:$$

Lemma 3. Let assumptions A1–A8 be valid and conjectures w_i^* be defined by (26)–(27). Then for any $G > 0$ and $Q \ge 0$ satisfying inequality (31), the following estimates hold:

$$\sum q_i'(G-0) \le \sum q_i'(G+0) \le \begin{cases} 1-\frac{2Q}{G}, & \text{if } I^+(G) \ne \emptyset, \\ 0, & \text{if } I^+(G) = \emptyset. \end{cases} \tag{32}$$

Proof. Since equality (30) takes place for $i \in I(G)$, we can use the Theorem of Implicit Function Differentiability and express the derivative

$$q_i'(G) = \frac{p'(1+\sigma_i) + \sigma_i p'' G + \alpha_i q_i p''}{f_i''(q_i) - \alpha_i p'}, \qquad i \in I(G). \tag{33}$$

Here the argument G is omitted in the righthand side. From A2 and A8, it follows that fracture (33) denominator always takes positive values. Therefore, the sign of derivative $q_i'(G)$ coincides with that of fraction (33) numerator.

It is readily verified that if $\alpha_i = 0$, then A2, (28) and $0 \le \sigma_i < 1$ imply

$$q_i'(G) = \frac{1}{f_i''(q_i)}[(1-\sigma_i)p' + \sigma_i(2p' + p''G)] < 0.$$

However, if $\alpha_i > 0$, the situation is more complex, since $q_i'(G)$ can take either positive or negative values . Suppose that $q_i'(G) > 0$. In this case, the numerator in (33) is obviously positive. Since $f_i'' \ge 0$, then deleting it we obtain from (33)

$$0 < q_i'(G) \le -\frac{1-\sigma_i}{\alpha_i} + \frac{2}{G}q_i(G) \le -1 + \frac{2}{G}q_i(G). \tag{34}$$

Thus we have shown that under inequality (31) holding (e.g. at an equilibrium point) there can be at most one subject with positive derivative $q_i'(G)$.

Now we can evaluate $\sum_{i \in I(G)} q_i'(G)$ in case when $\sum q_i(G) \le G - Q$. Indeed, if $I^+(G) = \emptyset$, then $\sum_{i \in I(G)} q_i'(G) \le 0$. Otherwise, if $I^+(G) = \{i_0\}$, then $q_{i_0} \le G - Q$, and (34) implies

$$\sum_{i \in I(G)} q_i'(G) \le q_{i_0}'(G) \le -1 + \frac{2q_{i_0}(G)}{G} \le 1 - \frac{2Q}{G}. \tag{35}$$

Since $q_i'(G_i + 0) = 0$ and $q_i'(G_i - 0) \le 0$ for $i \notin I(G)$, i.e. when $G \ge G_i$, the Lemma's assertion follows from inequality (35). ■

Theorem 4. Let assumptions A1–A8 be valid and conjectures w_i^* be defined by (26)–(27). In addition, if $Q = 0$, let $f_i''(q_i) > 0$ for all i and $q_i \ge 0$. Then the equilibrium vector $Z(Q)$ exists uniquely.

Proof. Existence of an equilibrium follows directly from Theorem 2. If $Q > 0$, then Lemma 3 implies $\sum_{i \in I(G)} q_i'(G) < 1$. Otherwise, i.e. if $Q = 0$, then the positivity of f_i'' allows one to re-write (34) with the strict inequality:

$$0 < q_i'(G) < -\frac{1-\sigma_i}{\alpha_i} + \frac{2}{G}q_i(G) \le -1 + \frac{2}{G}q_i(G),$$

that again leads to the estimate $\sum_{i \in I(G)} q_i'(G) < 1$. Therefore, in both cases, one can conclude that if balance equality (11) takes place for some quantity $G > 0$, then the difference $G - Q - \sum q_i(G)$ strictly increases with growing of G after that. So, the balance equality cannot hold for any G greater than the above quantity, which means that problem (11) – (13) has a unique solution. This completes the proof. ■

6 Influence Quotient as Function of Total Bargain Volume Only

In this section, we consider another particular case of the generalized Cournot model in which the agents' influence quotients are continuous functions of the total bargain volume only: $w_i = w_i(G)$, $i = 1, \dots, n$. In that case, the equilibrium problem reduces to the form: given $Q \ge 0$, find a n-tuple $[q_1(Q), \dots, q_n(Q)]$ such that, for each agent $i = 1, \dots, n$, under the assumption that $q_j(Q)$, $j \neq i$, are fixed, the value $q_i = q_i(Q) \ge 0$ solves the following (univariate) complementarity problem

$$\varphi_i(q_i) = f_i'(q_i) - q_i w_i(G) p'(G) - p(G) \ge 0; \tag{36}$$

$$\varphi_i(q_i) = 0, \qquad \text{if} \qquad q_i > 0; \tag{37}$$

here G is defined as the sum

$$G = Q + \sum q_i. \tag{38}$$

Complementarity problem (36)–(38) shows that the agent i need not vary its output q_i provided that he uses conjecture $w_i(G)$.

Begin with conditions that guarantee existence of solution to problem (36)–(38). Assume functions f_i, w_i and p to satisfy the following requirements.

AA1. Each of functions $f_i(q_i)$, $i = 1, \dots, n$, defined at $q_i \ge 0$ are convex, twice continuously differentiable and non-decreasing.

AA2. Inverse demand function $p(G)$ defined for positive values of G is twice continuously differentiable, takes positive values, and $p'(G) < 0$ for each $G > 0$.

AA3. For each $i = 1, \dots, n$ there exists a scalar $H_i > 0$ such that

$$f_i'(H_i) = p(H_i). \tag{39}$$

AA4. Each of the functions $w_i(G)$ defined at $G > 0$ are continuous and take non-negative values.

AA5. For every $i = 1, \dots, n$ and small enough $G > 0$, the following inequality holds:

$$f_i'(G) - \frac{1}{n} w_i(G) p'(G) G - p(G) < 0.$$

Remark that assumption AA5 is surely valid if the right-side derivative $p'(+0)$ is finite and assumptions AA1 – AA4 take place. Also notice condition (39) to be natural enough. Indeed, if the latter fails for some agent i, then one of the following relationships holds:

either

a) $f_i'(0) \ge \lim\limits_{G \to +0} p(G)$; in this case, the agent i produces nothing and can be excluded from the model;

or

b) $f_i'(G) < p(G)$ for every $G > 0$; moreover, if $\lim\limits_{G \to +\infty} [p(G) - f_i'(G)] > 0$, then production of an arbitrary large amount of the good is profitable for the i-th agent which means that problem (36)–(38) is unsolvable. Only in case when

$$\lim_{G \to +\infty} [p(G) - f_i'(G)] = 0 \quad \text{and} \quad \limsup_{G \to +\infty} [-p'(G)/p''(G)] < +\infty$$

the problem has a solution, but we do not consider that extremal case.

Since the problem considered here is a particular case of the generalized Cournot model examined in Sections 1 – 4, proofs of the theorems below are omitted.

Theorem 5 Let $Q > 0$ and assumptions AA1 – AA4 hold. Then problem (36)–(38) is solvable. In addition, if assumption AA5 is also valid, then the problem has a solution for $Q = 0$, too.

In order to establish the solution uniqueness, we keep assumptions AA1 – AA5 and add the following ones.

AA6. For each $i = 1, \ldots, n$, function w_i is differentiable and non-decreasing. Moreover, it takes only positive values not exceeding the unity.

AA7. The function $p(G)G$ is concave.

Theorem 6. If assumptions AA1 – AA7 are valid and $f_i''(G) > 0$ for every i and $G \ge 0$, then problem (36) – (38) is solvable uniquely.

Remark 5. In the particular case when $w_i \equiv 0$ for every i (the perfect competition), conditions AA1 – AA3 are sufficient for the existence of solution to problem (36) – (38). To guarantee the solution's uniqueness, we demand the derivative f_i' to increase strictly for each i.

7 Market with Leaders

In this Section, we generalize the Stackelberg model [14] and embed it into the above frameworks. In the classical Stackelberg model, one of the firms takes into account the response of the other market agents to its output variations, and maximizes its profit. In paper [11], such a firm is called a leader. Now we examine the case of several leaders, who act toward each other as the classical Cournot agents, whereas the followers behave according to the model of Section 4. As it was mentioned in the Introduction, we are going to embed the generalized Stackelberg model into the general framework making use of the first order optimality conditions. It means that we are looking for such a situation where the leaders' profit functions have stationary points. Unfortunately, the space limitations have not permitted to find out whether the stationary points satisfy the second order sufficient optimality conditions. So, from here onward, we will talk not about equilibria of the generalized Stackelberg model but of its stationary points.

Theorem 7. Let assumptions A1–A3, A5, A6, A8 hold, and functions $w_j^{\pm}(G; q_j)$, $j = s + 1, \dots, n$ be defined by equalities (26), (27). Then the generalized Stackelberg model has a stationary point.

Proof. Let the former s firms ($1 \le s < n$) be treated as leaders, i.e. they take into acccount the rest $(n - s)$ agents' reaction to variations of the leaders' total output $Q = \sum_{i=1}^{s} q_i$. Towards each other they act as the classical Cournot subjects, i.e. they suppose the other leaders not to change their outputs. At last, the latter $(n - s)$ firms use the above tactics with the extraneous supply Q and functions w_j^-, w_j^+ defined in (26)–(27). The classical Stackelberg model needs $s = 1$, $\alpha_i = 1$, $\sigma_i = 0$, $i = 2, \dots, n$. In view of A1–A8, for each $Q \ge 0$, solution $Z(Q) = [G(Q); q_{s+1}(Q); \dots; q_n(Q)]$ to problem (11)–(13) exists uniquely and is differentiable everywhere except for the points G_i at which only left- and right-side derivatives exist. Differentiating the balance equality

$$G = Q + \sum_{j=s+1} q_j(G(Q)) \tag{40}$$

by Q and using estimate (32), we show that $G(Q)$ is strictly increasing, $G(0) > 0$, and $G(Q) \ge Q$ for all $Q \ge 0$.

In order to embed our problem into the general model examined above, we need defining functions w_i^+ and w_i^- for $i = 1, \dots, s$. According to supposed leader's behaviour, the $i-$th leader's conjecture indicates that its output variation $(\eta - q_i)$ leads to the extraneous (for the second group of subjects) supply changing by the value $(\eta - q_i)$, too. Hence, the $i-$th leader assumes that the balance equality (36) remains valid after replacing Q by $(Q + \eta - q_i)$. Differentiating this equality by η from the left and right at the point $\eta = q_i$ and finding out the one-side derivatives $dG_i/d\eta(q_i \pm 0)$, we construct the functions w_i^+ and w_i^- for $i = 1, \dots, s$. Namely, if $G > G(0)$, then the value $Q(G) > 0$ (i.e. such that $G(Q(G)) = G$) is determined uniquely, and for $0 < q_i \le Q(G)$ we obtain

$$w_i^-(G; q_i) = \frac{1}{1 - \sum_{j=s+1} q_j'(G - 0)} = u^-(G); \tag{41}$$

$$w_i^+(G; q_i) = \frac{1}{1 - \sum_{j=s+1} q_j'(G + 0)} = u^+(G): \tag{42}$$

In view of Lemma 3,

$$0 < u^-(G) \le u^+(G) \le \max\left\{1; \frac{G}{2Q(G)}\right\}: \tag{43}$$

Furthermore, these functions coincide and are continuous everywhere except for the points G_i. As for the break points, function $u^-(G)$ is continuous from the left and $u^+(G)$ is continuous from the right at them. This is in line with assumption A4. In order to apply Theorem 1, we need defining formally functions $w_i^\pm$, $i = 1;\dots;s$ at all points $G > 0$, $0 < q_i \le G$ without violating condition A4.

Let us begin with the points $(G;q_i)$ for $G > G(0)$ and $Q(G) < q_i \le G$. To preserve the monotonicity of products $q_i w_i^\pm(G;q_i)$ and ensure condition (5) to hold at the considered points $(G;q_i)$, we put

$$w_i^\pm(G;q_i) = \alpha^\pm(G) + \frac{\sigma^\pm(G)}{q_i}; \tag{44}$$

having determined the quotients from the equations

$$\alpha^\pm(G) + \sigma^\pm(G)/Q(G) = u^\pm(G); \ \alpha^\pm(G) + \sigma^\pm(G)/G = 1: \tag{45}$$

Since $Q(G) < q_i \le G$, the values (44) lie between $u^\pm(G) > 0$ and 1, i.e. they are positive. Solving system (45) and substituting the values $\alpha^\pm$ and $\sigma^\pm$ into (44), we find

$$w_i^\pm(G;q_i) = \frac{G[1 - Q(G)/q_i] + [G/q_i - 1]Q(G)u^\pm(G)}{G - Q(G)}: \tag{46}$$

Since the quotient before $u^\pm(G)$ is non-negative for $q_i \le G$, thus defined functions $w_i^\pm$ keep the respective semi-continuity properties. Besides, in view of (39), the following upper limit value is finite and non-negative:

$$V^+ = \limsup_{G \to G(0)} \frac{Q(G)u^+(G)}{G - Q(G)} \le \frac{1}{2}: \tag{47}$$

Taking the upper limit in (46) when $G \to G(0)$, we put

$$w_i^+(G(0);q_i) = \left[1 - V^+\right] + \left[G(0)V^+\right]\Big/ q_i: \tag{48}$$

Non-negativity of (46) and (48) allows one to define $w_i^+(G;q_i) = 0$ for $G < G(0)$ and $w_i^-(G;q_i) = 0$ for $G \ge G(0)$.

It is readily verified that thus defined functions w_i^- and w_i^+ satisfy all the conditions of Theorem 1, and therefore, a stationary point does exists. The proof is complete. ■

Remark 6. Under a slightly stricter assumptions, it can be shown that $V^+ = 0$, hence $w_i^+(G(0);q_i) = 1$. For instance, to obtain that, it suffices to assume that either f_i'' are strictly positive, or $p(G)G$ is strictly concave, or at least two followers have positive outputs q_i at the equilibrium with $Q = 0$ (that is, in the absence of leaders).

8 Comparison of Equilibrium Totals in Cournot and Stackelberg Models

In this Section, we consider the classical Stackelberg model, i.e. we put $s = 1$. In that case we have $q_1 = Q$, and the leader maximizes his expected profit:

$$\max\{\mu_1(Q) = p(G(Q))\,Q - f_1(Q) \mid Q \ge 0\}; \tag{50}$$

considering the price $p(G(Q))$ as a function of his output Q. Here $G(Q) = Q + \sum_{j=2} q_j(Q)$, where $\sum_{j=2} q_j(Q)$ is the equilibrium total volume of outputs produced by the agents $j = 2, \ldots, n$, who solve problem (11)–(13) with the external supply Q and their influence quotients $w_j^{\pm}$ defined by (26)–(27).

In what follows we establish relationships between the solution Q^* of problem (50) and the equilibrium output volume $\hat{Q}$ of the first agent if he does not behave as a leader but uses, like others, the influence quotients $w_1^{\pm}$ defined by (26)–(27). Besides, it is interesting to compare the values Q^* and $\hat{Q}$ to the leader's optimal output volume $\bar{Q}$ when he ignores the variation of the price. In other words, he solves the following complementarity problem: find $\bar{Q} \ge 0$ such that

$$\beta_1(\bar{Q}) = f_1'(\bar{Q}) - p\Big(G(\bar{Q})\Big) \ge 0; \qquad \bar{Q} \cdot \beta_1(\bar{Q}) = 0. \tag{51}$$

Theorem 8. Let assumptions A1–A8 hold. Then

$$\max\{\hat{Q}; Q^*\} \le \bar{Q} \le H_1; \tag{52}$$

here $H_1 > 0$ is the scalar from assumption A3, i.e. such that $f_1'(H_1) = p(H_1)$.

Proof. In order to prove the inequality $Q^* \le \bar{Q}$, it suffices to demonstrate $\mu_1'(Q+0) < 0$ $\forall Q > \bar{Q}$. Assumptions A1, A2 and the inequality $G'(Q+0) > 0$ established above, imply the estimate

$$\mu_1'(Q+0) = p'(G(Q))\,G'(Q+0)Q - \beta_1(Q) < -\beta_1(Q) \qquad \forall Q > 0:$$

Definition of $\bar{Q}$ implies $\beta_1(\bar{Q}) \ge 0$. Therefore, it suffices to verify $\beta_1'(Q+0) > 0$ for every $Q > 0$. But we have $\beta_1'(Q+0) = f_1''(Q) - p'(G(Q))\,G'(Q+0)$, and the required inequality follows from assumptions A1–A2 and the property $G' > 0$. Thus the estimate $Q^* \le \bar{Q}$ is established.

Now verify the inequality $\hat{Q} \le \bar{Q}$. On the contrary, suppose that $\hat{Q} > \bar{Q}$. Since the function $G(Q)$ strictly increases, we have $G(\hat{Q}) > G(\bar{Q})$. According to the definition of $\hat{Q}$, we can write down

$$\chi(\hat{Q}) = p\Big(G(\hat{Q})\Big) + \alpha_1\hat{Q}p'\Big(G(\hat{Q})\Big)Q + \sigma_1 G(\hat{Q})p'\Big(G(\hat{Q})\Big) - f_1'(\hat{Q}) = 0:$$

Making use of conditions A1, A2, A8 and definition (51) of the value $\tilde{Q}$, put the series of inequalities

$$0 = \chi(\hat{Q}) < p\left(G(\tilde{Q})\right) - f_1'(\tilde{Q}) = -\beta_1(\tilde{Q}) \le 0;$$

that leads to the contradiction. Therefore, indeed $\hat{Q} \ge \tilde{Q}$, and the first inequality in series (52) is thus verified.

In order to establish the second inequality in (52), we remark that $f_1'(\tilde{Q}) \ge p\left(G(\tilde{Q})\right) \ge p(\tilde{Q})$ due to (51) and the function p being strictly decreasing. Now it follows from A1–A3 that $f_1'(Q) > p(Q)$ if $Q > H_1$. Therefore, $\tilde{Q} \le H_1$, and the theorem is proven completely. ■

Notice that the operation of taking the maximal of two values $\tilde{Q}$ and Q^* in estimate (52) is essential. The point is that in contrast to paper [11], where the estimate $\hat{Q} \ge Q^*$ has been established, the weaker conditions on the inverse demand curve $p(G)$ introduced in this paper do not allow one to deduce the similar estimate in the general case. Moreover, there exist such examples when the leadership of a weak (in a certain sense) agent leads to decreasing of his own (and therefore, the total) output level, i.e. $Q^* < \hat{Q}$ (and therefore, $G(Q^*) < G(\hat{Q})$). Now we will consider a particular case of our problem when more specific assumptions about the functions p and f_i permit us to prove the strict concavity of the leader's profit function $\mu_1(Q)$. After that, we will be able to establish the relationship between $\hat{Q}$ and Q^* using only a local information.

Namely, suppose the function p to be three times differentiable and satisfy the condition

$$p'' > 0 \qquad \text{and} \qquad \left(\frac{p''}{p'}\right)' \le -\frac{1}{G} \cdot \frac{p''}{p'} \tag{53}$$

for each $G > 0$. Moreover, let each cost function be linear, i.e. $f_i(q_i) = c_i q_i;\ i = 1, \dots, n$, and the influence quotients have the form (26)–(27) with $\alpha_i > 0$, $i = 1, \dots, n$. Now it is easily verified that

$$q_i(G) = \frac{c_i - p(G) - \sigma_i G p'(G)}{\alpha_i p'(G)}$$

for $i \in I = I(G) = \{1 \le i \le n \mid q_i(G) > 0\}$. This implies

$$q_i'(G) = -\frac{1 + \sigma_i}{\alpha_i} - \frac{\sigma_i G p''(G) + \alpha_i q_i(G) p''(G)}{\alpha_i p'(G)}; \qquad i \in I.$$

Thus, the equality yields

$$\sum_{i \in I} q_i'(G) = -\sum_{i \in I} \frac{1 + \sigma_i}{\alpha_i} - \frac{p''(G)}{p'(G)} \sum_{i \in I} q_i(G) - \frac{p''(G) G}{p'(G)} \sum_{i \in I} \frac{\sigma_i}{\alpha_i}.$$

So, at the point Q where the derivative $G'(Q)$ exists, it can be represented as follows

$$G'(Q) = \frac{1}{1 - \sum_{i \in I} q_i'(G(Q))} =$$

$$= \frac{1}{1+\sum_{i\in I}\frac{1+\sigma_i}{\alpha_i}+\frac{p''}{p'}\left[(G-Q)+G\sum_{i\in I}\frac{\sigma_i}{\alpha_i}\right]}. \tag{54}$$

Note that (54) and the concavity of the function $p(G)G$ imply the estimate

$$G'(Q) \geq \frac{1}{\sum_{i\in I}\frac{1-\sigma_i}{\alpha_i}-1-Q\frac{p''}{p'}} \geq \frac{1}{-Q\frac{p''}{p'}}. \tag{55}$$

Hence, $\pi_1'(Q) = p(G(Q)) + p'(G(Q))G'(Q)Q - c_1$, and

$$\pi_1''(Q) = 2p'(G(Q))G'(Q) + p''(G(Q))[G'(Q)]^2Q+$$

$$+p'(G(Q))G''(Q)Q. \tag{56}$$

By differentiating the terms in (54) we obtain (at the points Q where $G''(Q)$ exists) the following expression

$$G''(Q) = -(G')^2\left[\left(\frac{p''}{p'}\right)'(G-Q)G' + \frac{p''}{p'}(G'-1)\right].$$

Substituting the latter into (56) we have

$$\pi_1''(Q) = p'G'\left\{2 + 2Q\frac{p''}{p'}G' - Q\left[\frac{p''}{p'} + \left(\frac{p''}{p'}\right)'(G-Q)\right](G')^2\right\}.$$

Remark that the following inequality is implied by (53)

$$\frac{p''}{p'} + \left(\frac{p''}{p'}\right)'(G-Q) < 0.$$

Therefore, the second derivative $\pi_1''(Q)$ takes up negative values if $G' \geq \frac{1}{Q(-p''/p')}$. But the last inequality is valid by (53) for each $Q > 0$ such that the derivative $G'(Q)$ exists. At last, if only one-sided derivatives of the function G (and hence, of the function π_1) exist at the point Q, we will show that $\pi_1'(Q-0) > \pi_1'(Q+0)$. Indeed, if $i \in I(G-0)$ and $i \notin I(G+0)$, then it is readily verified that $q_i'(G-0) < 0$ and $q_i'(G+0) = 0$. Thus we deduce from (32) and the continuity of $G(Q)$ that $G'(Q-0) < G'(Q+0)$. Now making use of the formulae

$$\pi_1'(Q-0) = p(G(Q)) + p'(G(Q))G'(Q-0)Q - f_1'(Q);$$

$$\pi_1'(Q+0) = p(G(Q)) + p'(G(Q))G'(Q+0)Q - f_1'(Q)$$

and the function p being strictly decreasing, we obtain

$$\pi_1'(Q-0) > \pi_1'(Q+0)$$

for each $Q > 0$ where ν_1 is not differentiable.

Thus, the function ν_1 is strictly concave. Hence, we obtain the complete classification of comparison cases based upon the following local rules:

$$\text{(i) if } G'(\hat{Q}+0) \geq 1 \text{ and } G'(\hat{Q}-0) \leq 1; \text{ then } Q^* = \hat{Q}; \quad G(Q^*) = G(\hat{Q});$$

$$\text{(ii) if } G'(\hat{Q}+0) < 1; \text{ then } Q^* > \hat{Q}; \quad G(Q^*) > G(\hat{Q});$$

$$\text{(iii) if } G'(\hat{Q}-0) > 1; \text{ then } Q^* < \hat{Q}; \quad G(Q^*) < G(\hat{Q}).$$

For example, if $n = 1$ and $q_2'(\hat{G}) < 0$; then $Q^* > \hat{Q}$, consequently $G(Q^*) > G(\hat{Q})$. On the other hand, if $q_1'(\hat{G}) < 0$, and $q_2'(\hat{G}) > 0$, then $Q^* < \hat{Q}$, and hence, $G(Q^*) < G(\hat{Q})$ (that is, if the second agent produces more than one half of the total output at the Nash-Cournot equilibrium, then the leadership of the weaker agent 1 leads to decreasing of both his own and the total level of production).

9 Examples of Models

In this section, we consider oligopolistic models of different kinds considered in previous sections and compare their equilibria.

9.1 Comparison of Cournot Model and Model with High Expectations

Consider the Cournot model with three firms with linear costs at the market with a hyperbolic inverse demand function. In Example 1, we compare it with the generalized model in which all three agents assume $w_i \equiv 2$, $i = 1, 2$. It means that they are more precautious than agents of the standard Cournot model and conjecture almost the same behaviour of their rivals as their own.

Example 1 ($n \geq 2$).

We assume that

$$f_i(q_i) = c_i q_i; \qquad i = 1, \dots, n; \quad p(G) = AG^{-1}; \quad Q \geq 0.$$

Cournot Model

It means that

$$w_i(G; q_i) \equiv 1; \quad i = 1, \dots, n;$$

Equilibrum problem: find $(G; q_1, \dots, q_n) \geq 0$ such that

$$Q + \sum_{i=1} q_i = G; \tag{57}$$

$$\varphi_i \equiv c_i + \frac{A}{G^2} q_i - \frac{A}{G} \geq 0; \quad q_i \varphi_i = 0; \; i = 1, \ldots, n. \tag{58}$$

If $q_i > 0$, $i = 1, \ldots, n$, then

$$G^* = \frac{(n-1)A}{2\sum c_k} \left(1 + \sqrt{1 + 4Q \sum c_k / [A(n-1)^2]} \right);$$

If $Q = 0$ then

$$G^* = \frac{(n-1)A}{\sum c_k}; \tag{59}$$

$$q_i^* = \frac{(n-1)A}{(\sum c_k)^2} \left[\sum_{j \neq i} c_j - (n-2)c_i \right]; \quad i = 1, \ldots, n.$$

For instance, if $n = 3$ then (59) reduces to

$$G^* = \frac{2A}{\sum c_k};$$

and

$$q_i^* = \frac{2A}{(\sum c_k)^2} \left(\sum_{j \neq i} c_j - c_i \right); \quad i = 1, 2, 3;$$

For example, the net profit of agent $i = 1, 2, 3$ is expressed as

$$\pi_i^* = \frac{A}{(\sum c_k)^2} \sum_{j \neq i} c_j \left(\sum_{j \neq i} c_j - c_i \right).$$

Now we compare these values with the corresponding ones in case when $w_i(G, q_i) \equiv 2$, $i = 1, 2, 3$. Then

$$G^*|_{w_i \equiv 2} = \frac{A}{\sum c_k} < G^*|_{Cournot}.$$

The relationships between $q_i^*|_{w_i \equiv 2}$ and $q_i^*|_{Cournot}$ depend upon the distances between marginal cost values c_i, $c_i = 1, 2, 3$. More precisely, if all three really produce, then

$$q_i^*|_{w_i \equiv 2} \leq q_i^*|_{Cournot} \quad \text{if and only if} \quad c_i \leq \frac{3}{4} \sum_{j \neq i} c_j. \tag{60}$$

If $c_1 < c_2 < c_3$ then (60) implies that the strongest firm $i = 1$ decreases its output, whereas the two weaker ones may increase their outputs, in comparison to the classic Cournot equilibrium.

As for the profits,

$$\pi_i^*|_{w_i \equiv 2} = \frac{A}{2(\sum c_k)^2} \left(\sum_{j \neq i} c_j \right)^2 ;$$

which implies that

$$\pi_i^*|_{w_i \equiv 2} \geq \pi_i^*|_{Cournot}$$

if and only if

$$c_i \geq \frac{1}{2} \sum_{j \neq i} c_j :$$

For instance, if $c_1 < c_2 < c_3$ then we have

$$\pi_1^*|_{w_i \equiv 2} < \pi_1^*|_{Cournot}$$

for the strongest firm $i = 1$ but

$$\pi_3^*|_{w_i \equiv 2} > \pi_3^*|_{Cournot}$$

for the weakest firm $i = 3$.

9.2 Cournot Model vs Mixed Conjectures

In contrast to the previous example, in Example 2, the Cournot model is compared with the generalized one where each firm uses the mixed conjecture introduced in Section 5 with $\alpha_i = \sigma_i = 1/2$.

Example 2 (n=2).

Cournot Model with Two Agents

Finding an equilibrium, one comes to the following results:

$$G_C^* = \frac{A}{c_1 + c_2}; \quad q_1^* = \frac{Ac_2}{(c_1 + c_2)^2}; \quad q_2^* = \frac{Ac_1}{(c_1 + c_2)^2};$$

$$\pi_1^* = \frac{Ac_2^*}{(c_1 + c_2)^2}; \qquad \pi_2^* = \frac{Ac_1^2}{(c_1 + c_2)^2}:$$

Now we compare these values to those for the case

$$w_i(G; q_i) = \frac{1}{2} + \frac{1}{2} \cdot \frac{G}{q_i}; \quad i = 1; 2:$$

We get

$$G^* = \frac{A}{2(c_1 + c_2)} = \frac{1}{2} G_C^*; \quad q_i^* = \frac{1}{2} q_i^*|_{Cournot}; \; i = 1; 2;$$

but

$$\pi_i^* = \pi_i^*|_{Cournot} + \frac{Ac_1c_2}{2(c_1+c_2)^2} > \pi_i^*|_{Cournot}, \quad i = 1, 2.$$

In the other words, the mixed conjectures lead to lower outputs but higher profits for both agents.

9.3 Comparison of Stackelberg and Cournot Models

Here we illustrate the results of Section 8. Consider two firms with linear costs at the market with the hyperbolic inverse demand function,

$$n = 2;\ f_i(q_i) = c_i q_i;\ c_1 < c_2;\ Q = 0;\ p(G) = AG^{-1}.$$

At the Cournot equilibrium we have

$$q_i^0(\hat{G}) = \frac{c_j - c_i}{c_i + c_j}, \quad j \neq i;\ i = 1, 2.$$

Therefore, $q_1^0(\hat{G}) > 0$ (strong firm); $q_2^0(\hat{G}) < 0$ (weak firm).

Now we consider the following two cases.

1) Stakelberg Model: Strong Leader, Weak Follower

Doing the needed calculations we obtain

$$G^* = \begin{cases} A/(2c_1), & \text{if } c_2 \le 2c_1; \\ A/c_2, & \text{if } c_2 > 2c_1; \end{cases}$$

in both cases, $G^* > \hat{G}$;

$$q_1^* = \begin{cases} Ac_2/(4c_1^2), & \text{if } c_2 \le 2c_1; \\ A/c_2, & \text{if } c_2 > 2c_1; \end{cases}$$

in both cases, $q_1^* > \hat{q}_1$;

$$q_2^* = \begin{cases} A(2c_1 - c_2)/(4c_1^2), & \text{if } c_2 \le 2c_1; \\ 0, & \text{if } c_2 > 2c_1; \end{cases}$$

in both cases, $q_2^* < \hat{q}_2$;

$$\pi_1^* = \begin{cases} Ac_2/(4c_1), & \text{if } c_2 \le 2c_1; \\ A(c_2 - c_1)/c_2, & \text{if } c_2 > 2c_1; \end{cases}$$

in both cases, $\pi_1^* > \hat{\pi}_1$;

$$\pi_2^* = \begin{cases} A(2c_1 - c_2)^2/(4c_1^2); & \text{if } c_2 \le 2c_1; \\ 0; & \text{if } c_2 > 2c_1; \end{cases}$$

in both cases, $\pi_2^* < \hat{\pi}_2$.

That is, in the case of leadership of the stroger firm, both the total output and the individual outputs grow up, hence the price goes down, which is good for consumers. Moreover, the leader increases his profit in comparison to Cournot case, whereas the profit of the follower decreases.

2) Stackelberg Model: Weak Leader, Strong Follower

Again after calculations, one easily gets

$$G^* = \frac{A}{2c_2} < \frac{A}{c_1 + c_2} = \hat{G};$$

$$q_2^* = \frac{Ac_1}{4c_2^2} < \hat{q}_2;$$

$$q_1^* = \frac{A}{4c_2^2}(2c_2 - c_1) < \hat{q}_1;$$

$$\pi_2^* = \frac{Ac_1}{4c_2} > \hat{\pi}_2;$$

$$\pi_1^* = \frac{A}{4c_2^2}(2c_2 - c_1)^2 > \hat{\pi}_1:$$

Here we see that under leadership of the weaker firm, both the total output as well as the individual outputs fall down (which is bad for consumers as the price goes up!). As for the profits of the firms, they both gain more in comparison to the standard Cournot case.

9.4 Cournot Oligopoly vs Perfect Competition

In this subsection, we consider the homogeneous good market with two firms that have quadratic cost functions. We compare the standard Cournot oligopoly with the perfect competition case, i.e. when $w_i \equiv 0$, $i = 1; 2$.

Hence, we have

$$n = 2; \quad f_i(q_i) = c_i q_i^2; \ i = 1; 2; \quad p(G) = AG^{-1}; \ Q = 0:$$

1) Cournot Model.

Find $(G; q_1; q_2) \geq 0$ such that

$$q_1 + q_2 = G;$$

$$\varphi_i \equiv 2c_i q_i + \frac{A}{G^2} q_i - \frac{A}{G} \geq 0; \quad q_i \varphi_i = 0; \ i = 1; 2.$$

Then

$$G_C^* = \frac{\sqrt[4]{A}}{\sqrt[4]{4c_1c_2}};$$

$$q_1^* = \frac{\sqrt{A}}{2(\sqrt{c_1} + \sqrt{c_2})} \cdot \sqrt[4]{c_2/c_1}; \quad q_2^* = \frac{\sqrt{A}}{2(\sqrt{c_1} + \sqrt{c_2})} \cdot \sqrt[4]{c_1/c_2};$$

$$\pi_1^* = \frac{A}{2(\sqrt{c_1} + \sqrt{c_2})^2} (2c_2 + \sqrt{c_1c_2});$$

$$\pi_2^* = \frac{A}{2(\sqrt{c_1} + \sqrt{c_2})^2} (2c_1 + \sqrt{c_1c_2}).$$

2) Perfect Competition ($w_i \equiv 0$).

Find $(G; q_1; q_2) \geq 0$ such that

$$q_1 + q_2 = G;$$

$$\varphi_i \equiv 2c_i q_i - \frac{A}{G} \geq 0; \quad q_i \varphi_i = 0; \ i = 1; 2.$$

Then

$$G_P^* = \sqrt{\frac{A(c_1 + c_2)}{2c_1c_2}} > G_C^*;$$

As for the relationships between $q_{i;P}^*$ and q_i^*, it depends upon the relative value $\gamma = c_1/c_2$. The stronger agent always raises up its output, i.e. if $\gamma \geq 1$ then

$$q_{1;P}^* = \sqrt{\frac{Ac_2}{2c_1(c_1 + c_2)}} > q_1^*|_{\text{Cournot}};$$

whereas the output of the weaker agent can be less than that of the classic Cournot equilibrum, i.e.

$$q_{2;P}^* = \sqrt{\frac{Ac_1}{2c_2(c_1 + c_2)}} < q_2^*|_{\text{Cournot}};$$

if $0 < \gamma < \gamma^*$, where γ^* is the unique positive root of the cubic polynomial $y = x^3 - x^2 - x - 1$.

But for the profits we have

$$\pi^*_{1;P} = \frac{Ac_2}{2(c_1 + c_2)} < \pi^*_1;$$

$$\pi^*_{2;P} = \frac{Ac_1}{2(c_1 + c_2)} < \pi^*_2.$$

As we see, the perfect competition leads to higher total output but lower profits for both agents, in comparison to the standard Cournot model.

10 Conclusions

Results of the paper allow one to consider not only two separate models: Walras (perfect competition) and Cournot ones, but also the whole series of intermediate models determined by different values of the conjectural influence quotients $w_i \in [0; 1]$. Moreover, some brand new models of oligopoly appear when one allows the factors w_i to assume values greater than 1.

In addition, both classic and extended (to the case of several leaders) Stackelberg models can be embedded into the above-mentioned generalized oligopoly models by the explicit construction of appropriate influence functions $w_i(q_i; G)$ for the leaders.

Lastly, the influence quotients help one in conducting efficient comparative statics analysis for various models of oligopoly.

Acknowledgement

Authors would like to express their gratitude to the three anonymous references whose criticisms and valuable remarks have helped to improve the exposition.

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